Supersymmetric Sine-Gordon Model and the Eight-Vertex Free Fermion model with Boundary

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Abstract

We compute the boundary scattering amplitudes of the breathers of the supersymmetric sine-Gordon model using the fusion of the soliton-antisoliton pair scattering with the boundary with a known result of the soliton boundary scattering amplitudes. We also solve the boundary Yang-Baxter equation of the eight-vertex free fermion models to find the boundary reflection matrices. The former result is confirmed by the latter since the bulk S-matrices of the breathers can be identified with the trigonometric limit of the Boltzmann weights of the free fermion models. Our dual approach can answer a few quesions on the relationships between the free parameters in the boundary potential and those in the scattering amplitudes.

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1 Introduction and Motivation

The study of the two-dimensional integrable models of quantum field theories and statistical models based on the Yang-Baxter equation (YBE) has provided important understandings of nonperturbative aspects of these models and technical tools for applications to real physical problems. The YBE plays essential roles in establishing the integrability and solving the models. In the field theories, the YBE provides a consistency condition for the two-body scattering amplitudes (S-matrices) in the multi-particle scattering processes since the scattering is factorizable. With unitarity and crossing symmetry, the YBE can determine the S-matrix completely. In addition, the correlation functions may be obtained by computing multi-particle form-factors. The lattice models which are defined by the Boltzmann weights can have well-defined transfer matrices if they satisfy the YBE and can be diagonalized by independent technologies, such as the algebraic Bethe ansatz to extract exact properties of the model.

Recently there has been a lot of efforts in extending these approaches to models with boundaries. They are motivated by the fact that these models with the boundary have more applicability to real physical systems than those without one. For example, three-dimensional spherically symmetric physical systems can be effectively described on the half-line if s-wave element becomes dominant. One-channel Kondo problem, monopole-catalyzed proton decay are frequently cited examples. Also one can generalize the conventional periodic boundary condition of the statistical models to other types like the fixed and free conditions.

The existence of the boundary adds new quantities like boundary scattering amplitudes and Boltzmann weights, and one needs to extend the YBE to include these objects. The boundary Yang-Baxter equation (BYBE) (also known as the reflection equation) [1] plays the role of the YBE for the integrable statistical models [2, 3] and quantum field theories [4] in the presence of a boundary; it is the necessary condition for the integrability of these models.

The supersymmetric sine-Gordon model (SSG) preserves the integrability in the presence of the boundary [5] and the scattering amplitude of the SSG solitons with it has been computed [6]. In this paper we compute the boundary scattering amplitudes of the SSG breathers, the bound states of the soliton and antisoliton pair, in two independent ways. The first one is to consider the the eight-vertex free fermion models with the boundary and to solve the BYBE. This is related to the SSG model since the breather S-matrices are trigonometric limit of the free fermion models in certain regimes. The second is to compute the two-particle (a SSG soliton and an antisoliton) boundary scattering amplitudes and to take a limit of two rapidities so that they can form a bound state. This 'fusion procedure' can give independent check of our results. In addition, these two approaches can answer a few questions raised in the previous

study; how the supersymmetry of the SSG lagrangian with the boundary can be realized in the scattering matrix context and how the parameters in the lagrangian and those in the boundary scattering amplitudes are related.

2 SSG Breathers on a half line

The action of the SSG model is given by

$$S = \int dxdt \left[\frac{1}{2}(\partial_\mu \phi)^2 - i\overline{\psi}\ \partial\!\!\!/ \psi - \frac{m^2}{\beta^2}\cos^2\phi - 2m(\cos\frac{\beta\phi}{2})\overline{\psi}\psi \right], \tag{2.1}$$

where ϕ is a real scalar field and ψ is a Majorana fermion. β is a coupling constant and m is the mass parameter denoting the deviation from the massless theory. This theory has soliton spectrum $|K^{\pm}_{ab}(\theta)\rangle$ where 'ab' and '$\pm$' are the RSOS spins ($a, b = 0, \frac{1}{2}, 1$) and the topological charges ('$+$' for the soliton and '$-$' for the antisoliton), respectively and θ is the rapidity. The exact S-matrix of the SSG (anti-)solitons has the factorized form of [7]

$$S_{\rm SSG}(\theta) = S^{(4)}_{\rm RSG}(\theta) \otimes S_{\rm SG}(\theta).$$

The first S-matrix factor which acts on the supersymmetry (SUSY) charges is the RSOS S-matrix for the tricritical Ising model perturbed by the Φ_{13} operator;

$$S^{(4)}_{\rm RSG} = S^{ab}_{dc}(\theta) = U(\theta)\left(X^{ab}_{cd}\right)^{-\frac{\theta}{2\pi i}} \left[\sqrt{X^{ab}_{cd}}\sinh\left(\frac{\theta}{p}\right)\delta_{db} + \sinh\left(\frac{i\pi - \theta}{p}\right)\delta_{ac} \right], \tag{2.2}$$

for $|\mathbf{K}_{da}(\theta_1)\rangle + |\mathbf{K}_{ab}(\theta_2)\rangle \to |\mathbf{K}_{dc}(\theta_2)\rangle + |\mathbf{K}_{cb}(\theta_1)\rangle$ where $X^{ab}_{cd} = \left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]}\right)$ with q-number $[n] = (q^n - q^{-n})/(q - q^{-1})$ and $q = -e^{-i\pi/4}$ (Fig.1).

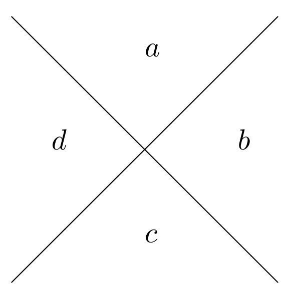

Fig. 1 Bulk S-matrix of the kinks

The second one is formally the sine-Gordon (SG) (anti)soliton S-matrix with the parameter given by $\gamma = 4\beta^2/(1-\beta^2/4\pi)$.[1] The factorized form of the S-matrix implies that the SSG soliton can be formally written as $|\mathbf{K}^{\pm}_{ab}(\theta)\rangle = |\mathbf{K}_{ab}(\theta)\rangle \otimes | \pm (\theta)\rangle$. The charge conjugation of the SSG solitons is defined by

$$C|\mathbf{K}^{\pm}_{ab}\rangle = |\mathbf{K}^{\mp}_{ba}\rangle. \tag{2.3}$$

For $n < 8\pi/\gamma \leq n+1$, the second factor, the SG S-matrix, has n poles in the physical strip corresponding to the SSG breathers. The threshold value of the SSG β to have any bound state is $\beta^2 = 4\pi/3$ compared with that of the SG, $\beta^2 = 4\pi$.

The bulk S-matrices of the breathers have been obtained by considering the residues of two solitons and two antisolitons scattering and taking appropriate limits on the rapidities. Due to the factorized form of the soliton S-matrix, the SSG breather S-matrices are also made up of two factors. The factors coming from the SG sector have been computed in [8] and they are completely diagonal since the masses of the SG breathers are non-degenerate. The second one comes from the four kinks scattering fusion processes of the RSG(4) [9]. It is obvious that this factor is non-diagonal since the SUSY makes the mass spectrum degenerate and the breathers form $N = 1$ supermultiplets.

Since two kinks can scatter only when they share a common RSOS spin, the two-kink states which form the SSG breathers can be written as

$$\begin{aligned}
|\psi^1_n(\theta)\rangle &= \frac{i\alpha_n}{\sqrt{2}}\left(|\mathbf{K}_{0\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}1}(\theta_2)\rangle - |\mathbf{K}_{1\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}0}(\theta_2)\rangle\right) , \\
|\phi^1_n(\theta)\rangle &= \frac{1}{\sqrt{2}}\left(|\mathbf{K}_{0\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}0}(\theta_2)\rangle + |\mathbf{K}_{1\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}1}(\theta_2)\rangle\right) , \\
|\psi^2_n(\theta)\rangle &= \frac{\alpha_n}{\sqrt{2}}\left(|\mathbf{K}_{\frac{1}{2}0}(\theta_1)\mathbf{K}_{0\frac{1}{2}}(\theta_2)\rangle - |\mathbf{K}_{\frac{1}{2}1}(\theta_1)\mathbf{K}_{1\frac{1}{2}}(\theta_2)\rangle\right) , \\
|\phi^2_n(\theta)\rangle &= \frac{1}{\sqrt{2}}\left(|\mathbf{K}_{\frac{1}{2}0}(\theta_1)\mathbf{K}_{0\frac{1}{2}}(\theta_2)\rangle + |\mathbf{K}_{\frac{1}{2}1}(\theta_1)\mathbf{K}_{1\frac{1}{2}}(\theta_2)\rangle\right)
\end{aligned} \tag{2.4}$$

where the rapidities are related as

$$\theta = \frac{1}{2}(\theta_1 + \theta_2) \ , \quad \theta_1 - \theta_2 = \triangle\theta_n$$

and

$$\alpha_n = \sqrt{\tan\left(\frac{\pi + i\triangle\theta_n}{4}\right)} \ , \quad \triangle\theta_n = i\pi - \frac{in\gamma}{8} \ ,$$

[1] Notice that the S-matrix of the SG model depends on $\gamma = \beta^2/(1-\beta^2/8\pi)$.

with $\triangle\theta_n$ corresponding to the mass pole of the breathers.

Notice that the bound states come in two sets distinguished by their superscripts $1, 2$ and only particles with the same superscripts can scatter. Each pair (ψ_n^a, ϕ_n^a) forms $N = 1$ supermultiplet (see Eq.(4.1)) and has the same bulk S-matrices, [9]

$$S(\theta) = \rho(\theta) \begin{pmatrix} \sin\frac{n\gamma}{16} + \frac{i}{2}\sinh\theta & & & -i\sin\frac{n\gamma}{16}\sinh\frac{\theta}{2} \\ & -\frac{i}{2}\sinh\theta & \sin\frac{n\gamma}{16}\cosh\frac{\theta}{2} & \\ & \sin\frac{n\gamma}{16}\cosh\frac{\theta}{2} & -\frac{i}{2}\sinh\theta & \\ -i\sin\frac{n\gamma}{16}\sinh\frac{\theta}{2} & & & \sin\frac{n\gamma}{16} - \frac{i}{2}\sinh\theta \end{pmatrix}, \tag{2.5}$$

where the column and row are arranged in the order of $\psi_n^a\psi_n^a$, $\psi_n^a\phi_n^a$, $\phi_n^a\psi_n^a$, $\phi_n^a\phi_n^a$ for $a = 1, 2$. The states in Eq.(2.4) are invariant under charge conjugation C, implying that they are real scalar particles and Majorana fermions. The least massive bound states ($n = 1$) are identified with the ψ and ϕ fields in the Lagrangian Eq.(2.1) and, indeed, the above S-matrix is identical to that obtained in [10] if we identify $\sin\frac{\gamma}{16}$ with f. Since there is only one fundamental field pair (ψ, ϕ) in the lagrangian, the two sets of the bound states which have the same S-matrices should be indentified, i.e. $|\psi_n^1(\theta)\rangle \equiv |\psi_n^2(\theta)\rangle$ and $|\phi_n^1(\theta)\rangle \equiv |\phi_n^2(\theta)\rangle$.

The function $\rho(\theta)$ satisfies the unitarity and crossing relations

$$\begin{aligned} &\rho(\theta)\rho(-\theta)(\sin^2\tfrac{n\gamma}{16} + \sinh^2\tfrac{\theta}{2})\cosh^2\tfrac{\theta}{2} = 1 \\ &\rho(\theta) = \rho(i\pi - \theta) \end{aligned} \quad . \tag{2.6}$$

The minimum solution to these equations have been given as [10]

$$\rho(\theta) = -\frac{2i}{\sinh\theta} Z(\theta) Z(i\pi - \theta) \tag{2.7}$$

where

$$\begin{aligned} Z(\theta) \quad = \quad & \frac{\Gamma(-i\theta/2\pi)}{\Gamma(1/2 - i\theta/2\pi)} \prod_{l=1}^{\infty} \left[\frac{\Gamma(n\gamma/16\pi - i\theta/2\pi + l)\Gamma(-n\gamma/16\pi - i\theta/2\pi + l - 1)}{\Gamma(n\gamma/16\pi - i\theta/2\pi + l + 1/2)\Gamma(-n\gamma/16\pi - i\theta/2\pi + l - 1/2)} \right. \\ & \left. \times \frac{\Gamma^2(-i\theta/2\pi + l - 1/2)}{\Gamma^2(-i\theta/2\pi + l - 1)} \right] . \end{aligned}$$

Now we introduce a boundary potential which preserves the integrability. The SSG boundary potential that gives conserved charges at the first order has been derived as [5]:

$$\mathcal{B}(\phi, \psi, \overline{\psi}) = \Lambda \cos\frac{\beta(\phi - \phi_0)}{2} + M\overline{\psi}\psi + \epsilon\psi + \overline{\epsilon}\overline{\psi}. \tag{2.8}$$

With the assumption of the complete integrability, one can use the BYBE to solve this model. For the purpose, we use an important property of the S-matrix in Eq.(2.5) that it satisfies the free fermion condition [11, 9]. Therefore it should be a special limit of the Boltzmann weights of the eight-vertex free fermion model given in the Appendix. Indeed, consider the regime $|h| > 1$ and take the following trigonometric limit:

$$k \to 0 \quad \text{with} \quad k \cosh \delta = k \sinh \delta \equiv -\frac{1}{\sin \frac{n\gamma}{16}} \; .$$

We obtain the breather S-matrix if $u \equiv -i\theta$. An immediate consequence of this observation is that the boundary S-matrix of the SSG breathers are given by the trigonometric limit of the boundary Boltzmann weights derived in the Appendix. The trigonometric limit of Eq.(5.17) becomes

$$R(\theta) = \mathcal{R}(\theta) \begin{pmatrix} \cosh \frac{\theta}{2} G_+(\theta) - i \sinh \frac{\theta}{2} G_-(\theta) & -i\epsilon \sinh \theta \\ -i \sinh \theta & \cosh \frac{\theta}{2} G_+(\theta) + i \sinh \frac{\theta}{2} G_-(\theta) \end{pmatrix} \tag{2.9}$$

where

$$G_\pm(\theta) = \alpha_\pm - \frac{\epsilon \alpha_\pm + \alpha_\mp}{\sin \frac{n\gamma}{16} - \epsilon} \sinh^2 \frac{\theta}{2} \tag{2.10}$$

and

$$\alpha_+^2 - \alpha_-^2 = 2 \left(\epsilon - \frac{1}{\sin \frac{n\gamma}{16}} \right) , \qquad \epsilon = \pm 1 \tag{2.11}$$

The overall factor $\rho(\theta), \mathcal{R}(\theta)$ are functions that ensure the unitarity and crossing symmetry of the S- and R- matrices, respectively.

The equations that determine $\mathcal{R}(\theta)$ are given by

$$\begin{aligned} &\mathcal{R}(\theta)\mathcal{R}(-\theta)(\cosh^2 \tfrac{\theta}{2} G_+^2(\theta) + \sinh^2 \tfrac{\theta}{2} G_-^2(\theta) + \epsilon \sinh^2 \theta) = 1 \\ &\rho(2\theta)\mathcal{R}(\tfrac{i\pi}{2} + \theta) \cosh \theta \left(\sin \tfrac{n\gamma}{16} - i\epsilon \sinh \theta\right) = \mathcal{R}(\tfrac{i\pi}{2} - \theta) \; . \end{aligned} \tag{2.12}$$

As usual we define

$$\mathcal{R}(\theta) = \mathcal{R}_0(\theta)\mathcal{R}_1(\theta)$$

such that

$$\begin{aligned} &\mathcal{R}_1(\theta) = \mathcal{R}_1(i\pi - \theta) \\ &\mathcal{R}_1(\theta)\mathcal{R}_1(-\theta) \left(c_0 + c_1 \sinh^2 \tfrac{\theta}{2} + c_2 \sinh^4 \tfrac{\theta}{4}\right) = 1 \end{aligned} \tag{2.13}$$

where

$$c_0 = \alpha_+^2 \; , \quad c_1 = \frac{\epsilon \left(\alpha_+ + \epsilon \alpha_-\right)^2}{\epsilon - \sin \frac{n\gamma}{16}} + 2\epsilon \; , \quad c_2 = \frac{\left(\alpha_+ + \epsilon \alpha_-\right)^2}{\left(\epsilon - \sin \frac{n\gamma}{16}\right)^2} \; ,$$

and

$$
\begin{aligned}
&\rho(2\theta)\mathcal{R}_0(\tfrac{i\pi}{2}+\theta)\cosh\theta\left(\sin\tfrac{n\gamma}{16}-i\epsilon\sinh\theta\right)=\mathcal{R}_0(\tfrac{i\pi}{2}-\theta)\\
&\mathcal{R}_0(\theta)\mathcal{R}_0(-\theta)\cosh\theta=1.
\end{aligned}
\tag{2.14}
$$

The factor $\mathcal{R}_1(\theta)$ carries information about the boundary conditions that are determined by the free parameters $\alpha_\pm$, and its minimum solution is given by

$$
\mathcal{R}_1(\theta)=\frac{1}{\alpha_+}\sigma(\chi,\theta)\sigma(\eta,\theta)
\tag{2.15}
$$

where the function $\sigma(\chi,\theta)$ is an infinite product of Γ function defined as

$$
\begin{aligned}
\sigma(\chi,\theta) &= \frac{\Pi(\chi,\pi/2+i\theta)\Pi(-\chi,\pi/2+i\theta)\Pi(\chi,-\pi/2-i\theta)\Pi(\chi,-\pi/2-i\theta)}{\Pi^2(\chi,\pi/2)\Pi^2(-\chi,\pi/2)}\ ,\\
\Pi(\chi,-i\theta) &= \prod_{l=1}^{\infty}\frac{\Gamma(l+\chi/\pi+i\theta/2\pi)}{\Gamma(l+1+\chi/\pi+i\theta/2\pi)}\ ,
\end{aligned}
$$

with the parameters χ,η defined by

$$
\cos^{-2}\chi+\cos^{-2}\eta=c_1/c_0\ ,\quad \cos^{-2}\chi\cos^{-2}\eta=c_2/c_0\ .
$$

The relations that determine $\mathcal{R}_0(\theta)$ can, similarly, be solved with minimum solution given by

$$
\mathcal{R}_0(\theta)=\frac{Y(\theta)Y(i\pi-\theta)}{\pi\sqrt{i\epsilon\sinh(2\theta)\ \rho(-\pi/2-2i\theta)}}
\tag{2.16}
$$

where

$$
Y(\theta)=\prod_{l=1}^{\infty}\frac{\Gamma(1-l+\epsilon n\gamma/36\pi+1/4+i\theta/2\pi)\Gamma(l-\epsilon n\gamma/36\pi-1/4-i\theta/2\pi)}{\Gamma(-l+\epsilon n\gamma/36\pi+1/4-i\theta/2\pi)\Gamma(l+1-\epsilon n\gamma/36\pi-1/4+i\theta/2\pi)}\ .
$$

3 Soliton Fusion

In [6], we studied the scattering theory of the SSG model on a half line based on its soliton states. Essentially, in the presence of a boundary, integrability of the SSG model requires that the boundary S-matrix of the solitons satisfy the BYBE, Eq.(5.2). We can assume naturally that this boundary S-matrix is also factorized into two parts:

$$R_{\rm SSG}(\theta) = R^{(4)}_{RSG}(\theta) \otimes R_{\rm SG}(\theta) \tag{3.1}$$

where $R_{\rm SG}$ and $R^{(4)}_{RSG}$ are the SG and RSOS(4) boundary scattering matrices, respectively. Writing it in this form, we can solve the BYBE separately. The SG part has been obtained in [4, 12] and the kinks part has been found in [6] to be of the form (Fig.2)

$$R^a_{bc}(\theta) = \mathcal{R}(\theta) \left(X^{bc}_{aa}\right)^{-\frac{\theta}{2\pi i}} \left[\delta_{b\neq c} X^a_{bc}(\theta) + \delta_{bc}\left(\delta_{b-1/2,a} U_a(\theta) + \delta_{b+1/2,a} D_a(\theta)\right)\right] \tag{3.2}$$

where a, b, c are the RSOS(4) spins.

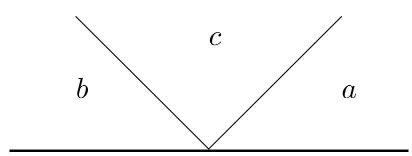

Fig.2 Boundary kink S-matrix

The explicit solutions are given by

$$X^{\frac{1}{2}}_{01} = s X^{\frac{1}{2}}_{10}\ , \quad \frac{U_{\frac{1}{2}}(\theta)}{X^{\frac{1}{2}}_{01}} = \frac{B}{\sinh\frac{\theta}{2}} + C\cosh\frac{\theta}{2}\ , \quad \frac{D_{\frac{1}{2}}(\theta)}{X^{\frac{1}{2}}_{01}} = \frac{B}{\sinh\frac{\theta}{2}} - C\cosh\frac{\theta}{2}\ ,$$
$$\frac{D_1(\theta)}{U_0(\theta)} = \frac{1 - A\sinh\frac{\theta}{2}}{1 + A\sinh\frac{\theta}{2}} \tag{3.3}$$

with A, B, C being the free parameters of the boundary, and the off-diagonal terms $X^{\frac{1}{2}}_{01}, X^{\frac{1}{2}}_{10}$ are independent of the spectral parameter and differ from each other by a gauge factor s. The overall function $\mathcal{R}(\theta)$ that guarantees boundary crossing and unitarity is given in [6].

As we have shown that the bound states of the RSG(4) kinks give rise to the $\psi_n(\theta), \phi_n(\theta)$ fields, whose bulk scattering matrix is given by Eq.(2.5), the scattering of these ψ_n, ϕ_n fields with the boundary is governed by the matrix given in Eq.(2.9). Hence we expect that by fusing the boundary scattering matrices of the kinks given above, we should reproduce Eq.(2.9). Before proceeding with the computation, it is worth recalling that, the fermionic and bosonic bound states come in two types; (ψ_n^1, ϕ_n^1) , (ψ_n^2, ϕ_n^2), which are, nevertheless, identified in the bulk since the scattering matrices of these two sets of particles have exactly the same form and are hence indistinguishable from each other. It would be natural to wonder whether the same holds true in the presence of a boundary. In fact, more intriguingly, notice that the boundary scattering matrices of the kinks carry more than one free parameters, while the scattering matrix given in Eq.(2.9) has only one. From the fusion equation, it is clear that the R-matrix of (ψ_n^2, ϕ_n^2), which will be built out of U_0, D_1, will contain one free parameter as in Eq.(2.9). However, that of (ψ_n^1, ϕ_n^1), which will be built from $X_{01}^{\frac{1}{2}}$, $X_{10}^{\frac{1}{2}}$, $U_{\frac{1}{2}}$, and $D_{\frac{1}{2}}$, will contain more than one free parameter, and this will be incompatible with Eq.(2.9) as they have the same bulk S-matrix given by Eq.(2.5). Also interesting is to try to clarify the relation of the two classes of solution, distinguished by $\epsilon = \pm 1$, with the fused boundary S-matrix of the kinks.

We represent the boundary scattering $|\mathbf{K}_{ab}(\theta_1)\mathbf{K}_{bc}(\theta_2)\rangle \longrightarrow |\mathbf{K}_{cd}(-\theta_1)\mathbf{K}_{de}(-\theta_2)\rangle$ in Fig.3.

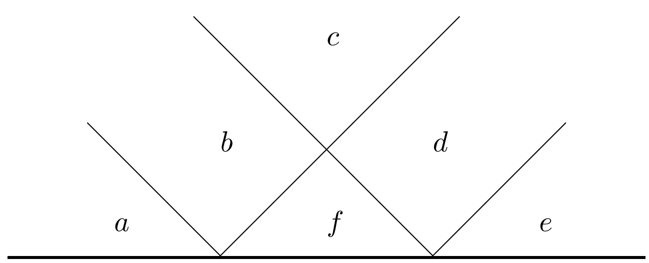

Fig.3 Boudnary scattering of the two-kink states

Let us begin with the bound states (ψ_n^2, ϕ_n^2). Using the fusion equations Eq.(2.4), we can construct their boundary S-matrix by combining U_0, D_1 and the bulk S-matrix

of the kinks as follows:

$$
\begin{aligned}
R^2_{\psi,\psi}(\theta) &= \tfrac{1}{2}\Big(U_0(\theta_1)U_0(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2) - U_0(\theta_1)D_1(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2) \\
&\quad + D_1(\theta_1)D_1(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2) - D_1(\theta_1)U_0(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2)\Big) \\
R^2_{\psi,\phi}(\theta) &= \tfrac{1}{2\alpha_n}\Big(U_0(\theta_1)U_0(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2) + D_1(\theta_1)U_0(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2) \\
&\quad - U_0(\theta_1)D_1(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2) - D_1(\theta_1)D_1(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2)\Big) \\
R^2_{\phi,\psi}(\theta) &= \tfrac{\alpha_n}{2}\Big(U_0(\theta_1)U_0(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2) + U_0(\theta_1)D_1(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2) \\
&\quad - D_1(\theta_1)U_0(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2) - D_1(\theta_1)D_1(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2)\Big) \\
R^2_{\phi,\phi}(\theta) &= \tfrac{1}{2}\Big(U_0(\theta_1)U_0(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2) + U_0(\theta_1)D_1(\theta_2)S^{0\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2) \\
&\quad + D_1(\theta_1)U_0(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}0}(\theta_1+\theta_2) + D_1(\theta_1)D_1(\theta_2)S^{1\frac{1}{2}}_{\frac{1}{2}1}(\theta_1+\theta_2)\Big)
\end{aligned}
\tag{3.4}
$$

where the superscript 2 on the boundary S-matrix refers to the second set of bound states.

As an example the explicit computations for $R^2_{\psi,\psi}(\theta)$ and $R^2_{\phi,\psi}$ look as follows:

$$
\begin{aligned}
R^2_{\psi,\psi}(\theta) &= \frac{U_0(\theta_1)U_0(\theta_2)}{\sqrt{2}\left(1+A\sinh\frac{\theta_1}{2}\right)\left(1+A\sinh\frac{\theta_2}{2}\right)}\left[\cosh\frac{\theta}{2}\left(1-A^2\sinh^2\frac{\triangle\theta_n}{4}+A^2\sinh^2\frac{\theta}{2}\right)\right. \\
&\quad \left. -i\sinh\frac{\theta}{2}\left(-1-A^2\sinh^2\frac{\triangle\theta_n}{4}+A^2\sinh^2\frac{\theta}{2}\right)\right] , \\
R^2_{\phi,\psi} &= \frac{i\alpha_1 A\sinh\theta U_0(\theta_1)U_0(\theta_2)}{\sqrt{2}\left(1+A\sinh\frac{\theta_1}{2}\right)\left(1+A\sinh\frac{\theta_2}{2}\right)}\left(\cosh\frac{\triangle\theta_n}{4}-i\sinh\frac{\triangle\theta_n}{4}\right) .
\end{aligned}
$$

Comparing them with the elements in the first column of the boundary S-matrix given in Eq.(2.9), and after dividing by an overall factor

$$
-\frac{\alpha_1 A U_0(\theta_1)U_0(\theta_2)}{\sqrt{2}\left(1+A\sinh\frac{\theta_1}{2}\right)\left(1+A\sinh\frac{\theta_2}{2}\right)}\left(\cosh\frac{\triangle\theta_n}{4}-i\sinh\frac{\triangle\theta_n}{4}\right) ,
$$

we see that the above $R^2_{\psi,\psi}$ has the form

$$
\cosh\frac{\theta}{2}G_+(\theta) - i\sinh\frac{\theta}{2}G_-(\theta) \tag{3.5}
$$

with

$$G_{\pm}(\theta) = -\left(\alpha_1 A\right)^{-1}\left(\cosh\frac{\triangle\theta_n}{4} - i\sinh\frac{\triangle\theta_n}{4}\right)^{-1}\left(\pm 1 - A^2\sinh^2\frac{\triangle\theta_n}{4} + A^2\sinh^2\frac{\theta}{2}\right) . \tag{3.6}$$

Comparing Eq.(3.6) with Eq.(2.10), we deduce that this fused S-matrix corresponds to the $\epsilon = 1$ case since the coefficients of $\sinh^2\frac{\theta}{2}$ are the same. Moreover, considering θ-independent terms in both equations, we find

$$\alpha_{\pm} = -\left(\alpha_1 A\right)^{-1}\left(\cosh\frac{\triangle\theta_n}{4} - i\sinh\frac{\triangle\theta_n}{4}\right)^{-1}\left(\pm 1 - A^2\sinh^2\frac{\triangle\theta_n}{4}\right) . \tag{3.7}$$

Notice that the coefficient of the $\sinh^2\frac{\theta}{2}$-term in Eq.(3.6) is given by

$$-\frac{\alpha_+ + \alpha_-}{\sin\frac{n\gamma}{16} - 1}.$$

Substituting the expressions for $\alpha_{\pm}$ into this, we can produce the corresponding coefficient in Eq.(2.10). In addition the $\alpha_{\pm}$ satisfy

$$\alpha_+^2 - \alpha_-^2 = 2\left(1 - \frac{1}{\sin\frac{n\gamma}{16}}\right) ,$$

which is consistent with Eq.(2.11). The same computations for the other two components $R^2_{\psi,\phi}, R^2_{\phi,\phi}$ also lead to the same conclusion as above. We therefore confirm that this fused boundary S-matrix indeed reproduces Eq.(2.9) with $\epsilon = 1$ and the mapping of the boundary parameter is given by

$$iA = \frac{\alpha_+ + \alpha_-}{1 - \left(\sin\frac{n\gamma}{16}\right)^{-1}} . \tag{3.8}$$

When (ψ_n^1, ϕ_n^1) scatter with the boundary, the out-states are linear combinations of (ψ_n^1, ϕ_n^1) and two other states. For example we have

$$\begin{aligned} R(\theta)|\phi_n^1(\theta)\rangle \;&=\; R^1_{\phi,\phi}(\theta)|\phi_n^1(-\theta)\rangle + R^1_{\psi,\phi}(\theta)|\psi_n^1(-\theta)\rangle \\ &+ BC\sinh\theta\cosh\frac{\triangle\theta_n}{2}\left(\cosh\frac{\theta}{2} - i\sinh\frac{\theta}{2}\right)|\Omega_1(-\theta)\rangle \\ &- \frac{1}{2}\sinh\theta\left(\cosh\frac{\triangle\theta_n}{4} + i\sinh\frac{\triangle\theta_n}{4}\right) \\ &\times\left[B\left(1+s\right) + i\frac{C}{2}\left(1-s\right)\left(\cosh\theta - \cosh\triangle\theta_n\right)\right]|\Omega_2(-\theta)\rangle , \end{aligned} \tag{3.9}$$

where $R^1_{\phi,\phi}, R^1_{\psi,\phi}$ are amplitudes to be given later. The two new states are given by

$$
\begin{aligned}
|\Omega_1(\theta)\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|\mathbf{K}_{0\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}0}(\theta_2)\rangle - |\mathbf{K}_{1\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}1}(\theta_2)\rangle\Big) \ , \\
|\Omega_2(\theta)\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|\mathbf{K}_{0\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}1}(\theta_2)\rangle + |\mathbf{K}_{1\frac{1}{2}}(\theta_1)\mathbf{K}_{\frac{1}{2}0}(\theta_2)\rangle\Big)
\end{aligned}
$$

which are orthogonal to (ψ^1_n, ϕ^1_n) states.

These new states do not scatter with (ψ^1_n, ϕ^1_n) at all in the bulk and their bulk S-matrix is different from Eq.(2.5). Moreover, the boundary potential in Eq.(2.8) contains only the (ψ, ϕ) pair and we should not expect any new particle to be created by the action of the boundary. Therefore, we should eliminate these extra states appearing in the fusion procedure so that the out-states are the linear combinations of only (ψ^1_n, ϕ^1_n). This is possible if we choose

$$
B = 0 \quad \text{and} \quad s = 1 \qquad \text{or} \qquad C = 0 \quad \text{and} \quad s = -1\ . \tag{3.10}
$$

The same conclusion is also arrived when the scattering of $|\psi^1_n(\theta)\rangle$ with the boundary is considered.

With the above restriction we can reproduce Eq.(2.9) by fusing the boundary and bulk S-matrices of the kinks as follows:

$$
\begin{aligned}
R^1_{\psi,\psi}(\theta) &= \tfrac{1}{2}\Big(U_{\frac{1}{2}}(\theta_1)U_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}0}_{1\frac{1}{2}}(\theta_1+\theta_2) + D_{\frac{1}{2}}(\theta_1)D_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}1}_{0\frac{1}{2}}(\theta_1+\theta_2) \\
&\quad + sS^{\frac{1}{2}1}_{1\frac{1}{2}}(\theta_1+\theta_2) + sS^{\frac{1}{2}0}_{0\frac{1}{2}}(\theta_1+\theta_2)\Big) \\
R^1_{\psi,\phi}(\theta) &= \tfrac{-i}{2\alpha_n}\Big(sD_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}1}_{0\frac{1}{2}}(\theta_1+\theta_2) + sU_{\frac{1}{2}}(\theta_1)S^{\frac{1}{2}1}_{1\frac{1}{2}}(\theta_1+\theta_2) \\
&\quad - D_{\frac{1}{2}}(\theta_1)S^{\frac{1}{2}0}_{0\frac{1}{2}}(\theta_1+\theta_2) - U_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}0}_{1\frac{1}{2}}(\theta_1+\theta_2)\Big) \\
R^1_{\phi,\psi}(\theta) &= \tfrac{i\alpha_n}{2}\Big(D_{\frac{1}{2}}(\theta_1)S^{\frac{1}{2}1}_{0\frac{1}{2}}(\theta_1+\theta_2) + U_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}1}_{1\frac{1}{2}}(\theta_1+\theta_2) \\
&\quad - sD_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}0}_{0\frac{1}{2}}(\theta_1+\theta_2) - sU_{\frac{1}{2}}(\theta_1)S^{\frac{1}{2}0}_{1\frac{1}{2}}(\theta_1+\theta_2)\Big) \\
R^1_{\phi,\phi}(\theta) &= \tfrac{1}{2}\Big(U_{\frac{1}{2}}(\theta_1)U_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}1}_{1\frac{1}{2}}(\theta_1+\theta_2) + D_{\frac{1}{2}}(\theta_1)D_{\frac{1}{2}}(\theta_2)S^{\frac{1}{2}0}_{0\frac{1}{2}}(\theta_1+\theta_2) \\
&\quad + sS^{\frac{1}{2}0}_{1\frac{1}{2}}(\theta_1+\theta_2) + sS^{\frac{1}{2}1}_{0\frac{1}{2}}(\theta_1+\theta_2)\Big)\ .
\end{aligned} \tag{3.11}
$$

Repeating the analysis given for $R^2_{\psi,\psi}$ before on the above amplitudes for the two cases, we can show that these scattering amplitudes coincide with that obtained in Eq.(2.9).

We can summarize these in the following mappings:

$$\begin{array}{lllll} \text{either} & B=0 & s=1 & C=\dfrac{-i(\alpha_+-\alpha_-)}{(\sin\frac{n\gamma}{16})^{\frac{1}{2}}-(\sin\frac{n\gamma}{16})^{-\frac{1}{2}}} & \epsilon=-1 \\ \text{or} & C=0 & s=-1 & B=\dfrac{(\sin\frac{n\gamma}{16})^{\frac{1}{2}}-(\sin\frac{n\gamma}{16})^{-\frac{1}{2}}}{\alpha_++\alpha_-} & \epsilon=1\ . \end{array} \tag{3.12}$$

The two classes ($\epsilon=\pm1$) of the solutions presented in Eq.(2.9) are indeed compatible with the boundary S-matrices obtained from the bound states.

Another aspect of the result is that the two $N=1$ supermultiplets distinguished by the superscripts $1,2$ scatter differently with the boundary since the scattering of (ψ_n^1,ϕ_n^1) is given by the S-matrices with $\epsilon=\pm1$, while that of (ψ_n^2,ϕ_n^2) by the S-matrix with $\epsilon=1$. Again we can require that the boundary potential should not add any new particle states in the theory and the identification of $|\psi_n^1\rangle\equiv|\psi_n^2\rangle$ and $|\phi_n^1\rangle\equiv|\phi_n^2\rangle$ made in the bulk should hold for the boundary as well. This dictates that the boundary scattering amplitudes for these two multiplets be identical. Thus we can find that

$$C=0, s=1, \quad \text{and} \quad iAB=\sqrt{\sin\frac{n\gamma}{16}}$$

with $\epsilon=1$, and there is only one free parameter associated with the boundary.

4 Boundary Supersymmetry

The SSG model has a supersymmetry and the bound states (ψ_n,ϕ_n) (we will drop the superscripts from now on) transform into each other under the action of the SUSY charges $Q,\overline{Q}$ in the following way [9]:

$$\begin{aligned} Q|\phi_n(\theta)\rangle &= \sqrt{im_n}\mathrm{e}^{\theta/2}|\psi_n(\theta)\rangle\ , & \overline{Q}|\phi_n(\theta)\rangle &= \sqrt{-im_n}\mathrm{e}^{-\theta/2}|\psi_n(\theta)\rangle\ , \\ Q|\psi_n(\theta)\rangle &= \sqrt{-im_n}\mathrm{e}^{\theta/2}|\phi_n(\theta)\rangle\ , & \overline{Q}|\psi_n(\theta)\rangle &= \sqrt{im_n}\mathrm{e}^{-\theta/2}|\phi_n(\theta)\rangle\ , \end{aligned} \tag{4.1}$$

where $m_n=2\sin\frac{n\gamma}{16}$ is the mass of the n-th bound state and we have not included the bound states of the SG sector since the SUSY charges act trivially on them.

One can show that the bulk S-matrix of the bound states is invariant under the action of the SUSY charges, namely,

$$S_{12}(\theta)Q_{12}(\theta)=Q_{21}(-\theta)S_{12}(\theta)\ , \qquad S_{12}(\theta)\overline{Q}_{12}(\theta)=\overline{Q}_{21}(-\theta)S_{12}(\theta)\ . \tag{4.2}$$

It is interesting to examine the action of these SUSY charges on the boundary S-matrix to see whether SUSY can be maintained with the presence of the boundary. In fact, it

is argued that one can retain only 'half' of the supersymmetry $Q \pm \overline{Q}$ in the presence of a boundary [13]. We shall see that this is the case for the SSG model.

One can write the SUSY charges Q and $\overline{Q}$ as 2×2 matrices when acting on one-particle state as

$$
\begin{aligned}
Q &= \begin{pmatrix} 0 & e^{\frac{\theta}{2}-\frac{i\pi}{4}} \\ e^{\frac{\theta}{2}+\frac{i\pi}{4}} & 0 \end{pmatrix} , \\
\overline{Q} &= \begin{pmatrix} 0 & e^{-\frac{\theta}{2}+\frac{i\pi}{4}} \\ e^{-\frac{\theta}{2}-\frac{i\pi}{4}} & 0 \end{pmatrix}
\end{aligned}
\tag{4.3}
$$

where we have arranged the basis in the order of ψ_n, ϕ_n.

Consider the action of the linear combination of charges $Q(\theta)+\beta\overline{Q}(\theta)$ on the boundary S-matrix:

$$R(\theta)\left[Q(\theta)+\beta\overline{Q}(\theta)\right]-\left[Q(-\theta)+\beta\overline{Q}(-\theta)\right]R(\theta)\ .$$

Using Eq.(2.9), we deduce that for the above to vanish, we have either

$$\beta = 1 \quad \text{for} \quad \alpha_+ + \alpha_- = 0 \tag{4.4}$$

or

$$\beta = -1 \quad \text{for} \quad \alpha_+ - \alpha_- = 0\ . \tag{4.5}$$

From Eq.(2.11) we see that for the relation $\alpha_+ = \pm\alpha_-$ to be true, we must have $|\alpha_\pm| \to \infty$. In this limit, the boundary S-matrix becomes diagonal

$$R(\theta) = \mathcal{R}_0(\theta) \begin{pmatrix} \cosh\frac{\theta}{2} + i\beta\sinh\frac{\theta}{2} & 0 \\ 0 & \cosh\frac{\theta}{2} - i\beta\sinh\frac{\theta}{2} \end{pmatrix} \tag{4.6}$$

and there is no boundary free parameter left, a conclusion which is in agreement with [5] that uses a different approach.

We would like to comment that this SUSY preserving boundary S-matrix, when regarded as the boundary reflection matrix of a lattice model, is the one that gives rise to a spin chain that possesses the quantum group $U_{1,q^2}gl(1|1)$ symmetry.

5 Discussion

In this paper we considered the SSG model with boundary from two points of view. One is to consider the BYBE of the SSG breathers which are related to the eight-vertex free fermion model and the other is to use fusion of two kinks which are related to RSOS(4) model. By matching the boundary S-matrices obtained from these two approaches and requiring that there is only one $N = 1$ supermultiplet for the bound

states, we reduced the four parameters s, A, B, C in the SUSY sector of the SSG soliton scattering with the boundary to one, and also constrained the solutions of the BYBE to the free fermion model to that with $\epsilon = 1$.

If we change β to $i\beta$ in the SSG model, we get the supersymmetric sinh-Gordon model with the difference that (ψ_1, ϕ_1) are the only particles which are not soliton-antisoliton bound states as there are no solitons (anti-solitons) in the theory. Therefore the above argument for the sine-Gordon theory does not apply here and both of the $\epsilon = \pm 1$ solutions are allowed. Since these solutions are mutually exclusive and associated with boundary potentials, they must be two different classes of the boundary potential which preserve integrability. In addition, each of these potentials should have at least one boundary coupling parameter which is related to that in the boundary S-matrix.

The SSG boundary potential in Eq.2.8, derived from the condition that there exist the conserved charges at the first order, contains five parameters. If we compare this with the SSG boundary soliton S-matrix, which has only three parameters (two from the sine-Gordon [14] and one from the SUSY sector), we can think of two possibilities; One is that two of the five parameters in Eq.(2.8) should disappear when higher order conserved charges are constructed. The other possibility is that all the five parameters survive and the soliton boundary S-matrix introduce two more parameters in its overall CDD factor.

Furthermore, boundary SUSY is realized only when the five parameters are fixed as;

$$\Lambda = \pm 8, M = \pm 1, \phi_0 = 0, \epsilon = \overline{\epsilon} = 0. \tag{5.1}$$

So there is no free parameters and the $\pm$ are related to the "half" SUSY $Q \pm \overline{Q}$ respectively. For the boundary S-matrix, we found that indeed only "half" SUSY can be realized. In which case, the boundary S-matrix becomes diagonal given in Eq.(4.6) without any parameters, but it also comes in two classes which possess the "half" SUSY respectively. This means there is no off-diagonal $(\phi \to \psi, \psi \to \phi)$ scattering amplitude and can be understood from the fact that the boundary potential deos not includes any fermion number violating term as far as $\epsilon = \overline{\epsilon} = 0$. This SUSY, however, acts trivially on the sine-Gordon soliton sector, which contains two free parameters, this leads to the conclusion that at the SUSY points these two parameters become unphysical.

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Note Added

After we completed and submitted our paper, we have noticed that [22] obtained a similar result as ours on the boundary supersymmetry covered in sect.4. We thank the authors for the information.

Appendix:

The Eight-Vertex Free Fermion Model with Boundary

We present solutions to the boundary Yang-Baxter equation for the general eight-vertex free fermion model here.

Recall the boundary Yang-Baxter equation takes the form [1, 2]

$$K_1(u_1)R_{12}(u_1+u_2)K_2(u_2)R_{21}(u_2-u_1) = R_{12}(u_2-u_1)K_2(u_2)R_{21}(u_1+u_2)K_1(u_1) \quad (5.2)$$

where $R(u)$ and $K(u)$ are, respectively, the bulk and boundary R-matrices.

The eight-vertex free fermion model has been studied by a number of authors [11, 15, 16, 17, 18, 19] the bulk R-matrix takes the form

$$R(u) = \begin{pmatrix} a_+ & & & d \\ & b_+ & c & \\ & c & b_- & \\ d & & & a_- \end{pmatrix} \quad (5.3)$$

where $a_\pm, b_\pm, c, d$ denote the usual vertex weights that depend on the spectral parameter u and other parameters of the model. These weights satisfy the free fermion condition

$$a_+a_- + b_+b_- - c^2 - d^2 = 0 \quad (5.4)$$

and R-matrices with the same parameters Γ and h given by

$$\Gamma = \frac{2cd}{a_+b_- + a_-b_+} \qquad h = \frac{a_-^2 + b_+^2 - a_+^2 - b_-^2}{2\,(a_+b_- + a_-b_+)} \quad (5.5)$$

commute.

The extreme anisotropic limit of the bulk R-matrix commutes with the quantum spin chain with local Hamiltonian given by

$$\mathcal{H}_{j,j+1} = \sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+ + \Gamma\left(\sigma_j^+\sigma_j^+ + \sigma_j^-\sigma_{j+1}^-\right) - \frac{h}{2}\left(\sigma_j^z + \sigma_{j+1}^z\right) \, . \quad (5.6)$$

Thus h can be interpreted as the bulk magnetic field of the spin chain. Using duality transformation or other argument, one can show that critical points must occur for $|h| = 1$. Here we shall adopt the parameterization given in [15]; For $|h| < 1$ the vertex weights are given by

$$\begin{array}{lcl} a_\pm & = & \cosh\gamma \,\mathrm{cn}\,\frac{u}{2} \mp \sinh\delta \,\mathrm{sn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} \\ b_\pm & = & \cosh\delta \,\mathrm{sn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} \pm \sinh\gamma \,\mathrm{cn}\,\frac{u}{2} \\ c & = & \mathrm{dn}\,\frac{u}{2} \\ d & = & k\,\mathrm{sn}\,\frac{u}{2}\,\mathrm{cn}\,\frac{u}{2} \end{array} \quad (5.7)$$

with

$$\Gamma = \frac{k}{\cosh(\gamma+\delta)} \ , \qquad h = \tanh(\gamma+\delta) \tag{5.8}$$

where k is the modulus of the elliptic functions. While for $|h| > 1$ [2]

$$\begin{array}{lcl} a_{\pm} & = & \cosh\gamma \ \mathrm{dn}\,\frac{u}{2} \pm k\cosh\delta \ \mathrm{sn}\,\frac{u}{2} \ \mathrm{cn}\,\frac{u}{2} \\ b_{\pm} & = & \pm\sinh\gamma \ \mathrm{dn}\,\frac{u}{2} - k\sinh\delta \ \mathrm{sn}\,\frac{u}{2} \ \mathrm{cn}\,\frac{u}{2} \\ c & = & \mathrm{cn}\,\frac{u}{2} \ \mathrm{dn}\,\frac{u}{2} \\ d & = & k' \ \mathrm{sn}\,\frac{u}{2} \end{array} \tag{5.9}$$

with

$$\Gamma = \frac{k'}{k\sinh(\gamma+\delta)} \ , \qquad h = \coth(\gamma+\delta) \tag{5.10}$$

where k' is the complementary modulus of the elliptic functions.

These R-matrices satisfy the unitarity and cross-unitarity properties

$$R(u)R(-u) \propto \mathbf{1} \ , \qquad R(u)^{\mathrm{t}_1} R(2K-u)^{\mathrm{t}_2} \propto \mathbf{1} \ . \tag{5.11}$$

So the crossing parameter is $2K$ a half-period of the elliptic functions. In general, due to the asymmetry, the R-matrices do not have crossing symmetry. For the regime $|h| < 1$, when $\gamma = \delta$, we have

$$R(2K-u)^{\mathrm{t}_2} = \sigma_1^x R(u) \sigma_1^x \ , \tag{5.12}$$

hence the basis of the $\mathbf{C}^2$ space can be given an interpretation of up-down spins. As for the regime $|h| > 1$, when $\gamma = 0$, we have instead

$$R(2K-u)^{\mathrm{t}_1} = R(u) \ . \tag{5.13}$$

A natural interpretation of this property can be given if we regard the basis of the $\mathbf{C}^2$ space as two distinct degrees of freedom such as a boson and a fermion.

Note also that for $\gamma = 0$, these R-matrices are regular in that

$$R_{12}(0) \propto \mathcal{P}_{12} \tag{5.14}$$

where $\mathcal{P}_{12}$ is the exchange operator of the spaces $1, 2$.

With the bulk R-matrix given, it is straight forward to solve for the boundary K-matrix using Eq.(5.2). We again find that in order for solutions to exist, the bulk parameter γ has to vanish, essentially this is due to the asymmetry of the vertex weights $b_{\pm}$. With this restriction, the most general solution is given as follows:

[2] The spectral parameter here differs from that in [15] by a shift of K.

For $|h| < 1$,

$$K(u) = \begin{pmatrix} G_+(u)\,\mathrm{cn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} - G_-(u)\,\mathrm{sn}\,\frac{u}{2} & 2\epsilon\,\mathrm{sn}\,\frac{u}{2}\,\mathrm{cn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} \\ 2\,\mathrm{sn}\,\frac{u}{2}\,\mathrm{cn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} & G_+(u)\,\mathrm{cn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} + G_-(u)\,\mathrm{sn}\,\frac{u}{2} \end{pmatrix} \tag{5.15}$$

where

$$\begin{aligned} G_+ &= \alpha_+ + \frac{k\left((1-\epsilon k\cosh\delta)\,\alpha_+ + \sinh\delta\alpha_-\right)}{\epsilon\cosh\delta - k}\,\mathrm{sn}^2\frac{u}{2} \\ G_- &= \alpha_- + \frac{k\left((1-\epsilon k\cosh\delta)\,\alpha_- - k'^2\sinh\delta\alpha_+\right)}{\epsilon\cosh\delta - k}\,\mathrm{sn}^2\frac{u}{2} \end{aligned} .$$

Here $\alpha_\pm$ are free parameters associated with the boundary that are related by

$$k'^2\alpha_+^2 + \alpha_-^2 = 2\left(k^{-1}\cosh\delta - \epsilon\right) \tag{5.16}$$

and $\epsilon = \pm 1$.

For $|h| > 1$,

$$K(u) = \begin{pmatrix} G_+(u)\,\mathrm{cn}\,\frac{u}{2} + G_-(u)\,\mathrm{sn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} & 2\epsilon\,\mathrm{sn}\,\frac{u}{2}\,\mathrm{cn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} \\ 2\,\mathrm{sn}\,\frac{u}{2}\,\mathrm{cn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} & G_+(u)\,\mathrm{cn}\,\frac{u}{2} - G_-(u)\,\mathrm{sn}\,\frac{u}{2}\,\mathrm{dn}\,\frac{u}{2} \end{pmatrix} \tag{5.17}$$

where

$$G_\pm(u) = \alpha_\pm - \mathrm{sn}^2\frac{u}{2}\,\frac{\alpha_\pm\epsilon k\sinh\delta + \alpha_\mp k' k\cosh\delta}{k' + \epsilon k\sinh\delta} .$$

Again $\alpha_\pm$ are free parameters associated with the boundary that are related by

$$\alpha_+^2 - \alpha_-^2 = 2\epsilon\left(1 + \frac{\epsilon k\sinh\delta}{k'}\right)$$

and $\epsilon = \pm 1$.

Hence for each regime there are two classes of solution distinguished by $\epsilon = \pm 1$ with one independent parameter.

Below we shall list a number of properties of these K-matrices.

1. Regularity

$$K(0) \propto \mathbf{1} . \tag{5.18}$$

2. Unitarity

$$K(u)K(-u) \propto \mathbf{1} \ . \tag{5.19}$$

3. Corresponding Quantum Spin Chains

   With the boundary K-matrix and the bulk R-matrix, one can construct an integrable quantum spin chain with fixed boundary condition using the prescription given in [2]. The quantity that generates the spin chain Hamiltonian and other commuting conserved charges is

$$t(u) = \mathrm{tr}\left(K(K-u;\alpha)T(u)K(u;\tilde{\alpha})T^{-1}(-u)\right)$$

   where $T(u)$ is the usual bulk monodromy matrix constructed out of the R-matrix, and $\alpha, \tilde{\alpha}$ denote, respectively, the parameters $\alpha_\pm, \tilde{\alpha}_\pm$ that are associated with the two boundaries of the spin chain. A new feature of the model is that $\mathrm{tr}\,(K(K;\tilde{\alpha}))$ vanishes, as a result we have

$$t(u) = (\text{const.})u\mathbf{1} + (\text{const.})u^2\mathcal{H} + \cdots \ ,$$

   which is common in models with a supersymmetry.[3] In the $|h| < 1$ regime, we find

$$\begin{aligned}
\mathcal{H} \ \propto \ & \sqrt{\cosh^2\delta - k^2}\sum_{j=1}^{N-1}\left[\sqrt{\frac{\cosh\delta + k}{\cosh\delta - k}}\sigma_j^x\sigma_{j+1}^x + \sqrt{\frac{\cosh\delta - k}{\cosh\delta + k}}\sigma_j^y\sigma_{j+1}^y \right.\\
& \left. -\sqrt{\frac{\sinh\delta + ik'}{\sinh\delta - ik'}}\sigma_j^z - \sqrt{\frac{\sinh\delta - ik'}{\sinh\delta + ik'}}\sigma_{j+1}^z\right] + \left(ik' - \frac{\tilde{\alpha}_-}{\tilde{\alpha}_+}\right)\sigma_1^z \\
& + \frac{2}{\tilde{\alpha_+}}\left(\sigma_1^- + \epsilon\sigma_1^+\right) + \frac{1 - \epsilon k\cosh\delta - ik'\sinh\delta}{1 - \epsilon k\cosh\delta + \frac{\alpha_-}{\alpha_+}\sinh\delta}\left(\frac{\alpha_-}{\alpha_+} - ik'\right)\sigma_N^z \\
& + \frac{2\left(k + \epsilon\cosh\delta\right)}{\alpha_+\left(1 - \epsilon k\cosh\delta\right) + \alpha_-\sinh\delta}\left(\epsilon\sigma_N^- + \sigma_N^+\right) \ .
\end{aligned} \tag{5.20}$$

   While for $|h| > 1$, we find

$$\begin{aligned}
\mathcal{H} \ \propto \ & -\sqrt{k^2\cosh^\delta - 1}\sum_{j=1}^{N-1}\left[\sqrt{\frac{k\sinh\delta - k'}{k\sinh\delta + k'}}\sigma_j^x\sigma_{j+1}^x + \sqrt{\frac{k\sinh\delta + k'}{k\sinh\delta - k'}}\sigma_j^y\sigma_{j+1}^y \right.\\
& \left. -\sqrt{\frac{k\cosh\delta + 1}{k\cosh\delta - 1}}\sigma_j^z - \sqrt{\frac{k\cosh\delta - 1}{k\cosh\delta + 1}}\sigma_{j+1}^z\right] + \left(\frac{\tilde{\alpha}_-}{\tilde{\alpha}_+} - 1\right)\sigma_1^z \\
& + \frac{2}{\tilde{\alpha}_+}\left(\sigma_1^- + \epsilon\sigma_1^+\right) + \left(\frac{\alpha_-}{\alpha_+} + 1\right)\sigma_N^z + \frac{2}{\alpha_+}\left(\epsilon\sigma_N^- + \sigma_N^+\right) \ .
\end{aligned} \tag{5.21}$$

[3] The same property is also observed in [17] for a special case of this model.

4. Quantum Group symmetry

   These spin chains include, as a special case, that obtained by [17] which has the quantum group $U_{p,q}(gl(1|1))$ (or $CH_q(2)$) symmetry where p, q are related to k, δ (see [17, 18, 19]). This can be seen by taking the limit $|\alpha_+|, |\alpha_-|$ $(|\tilde{\alpha}_+|, |\tilde{\alpha}_-|) \to \infty$, which, from the relation satisfied by them, implies that the ratio of them is finite. For the regime $|h| < 1$, taking $\frac{\tilde{\alpha}_-}{\tilde{\alpha}_+} = \frac{\alpha_-}{\alpha_+} = \pm ik'$ gives the quantum group symmetric spin chain. While for $|h| > 1$, we have to take $\frac{\tilde{\alpha}_-}{\tilde{\alpha}_+} = -\frac{\alpha_-}{\alpha_+} = \pm 1$. Note that in this limit the K-matrices are diagonal.

5. Symmetric Limit

   It is well known that in the symmetric limit $\delta = 0$, (in addition to $\gamma = 0$) the R-matrix for the regime $|h| < 1$ is a special case of Baxter's symmetric eight-vertex model. Therefore, in the symmetric limit, the K-matrix in the $|h| < 1$ regime should also be a special case of that obtained in [20, 21]. Recall that there are three boundary free parameters ξ, λ, μ there, but we have only one free parameter here. We find that indeed our solution corresponds to the case where one of the above three parameters vanishes with the remaining two related. It is intriguing that this way of approaching the symmetric free fermion gives rise to K matrices of less boundary free parameters.

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